Applications of the compensated compactness method on hyperbolic conservation systems

Yunguang Lu

Department of Mathematics

National University of Colombia

e-mail:ylu@unal.edu.co

ALAMMI 2009

In this talk, I would like to introduce the applications of the compensated compactness method on hyperbolic conservation systems of two equations.

We consider the following nonlinear system

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \tag{1}$$

where u and v are in R. We let $U = (u,v)$ and $F(U) = (f,g)$ so that the equations in (1) can be written as

$$U_t + dF(U)U_x = 0, \tag{2}$$

where $dF(U)$ is the Jacobian matrix of F.

We say that system (2) is hyperbolic if dF has two real eigenvalues $\lambda_1$and λ_2. System (2) is called strictly hyperbolic if λ_1 and λ_2 are distinct, i.e., $\lambda_1 < \lambda_2$. If λ_1 and λ_2 coincide at some points or domains, system (2) is called nonstrictly hyperbolic or hyperbolically degenerate.

It is well-known that the solutions for the Cauchy problem of nonlinear hyperbolic system (1) with bounded measurable initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)). \quad (3)$$

always have singularity (Discontinuity or Shock Waves) at finite time even if the initial data is small and smooth. We have to study the generalized solutions in the sense of distributions:

A pair of bounded functions $(u(x,t), v(x,t))$ is called a generalized solution of the Cauchy problem $(1),(3)$ if

$$\begin{cases} \iint\limits_{t>0} (u\phi_t + f(u,v)\phi_x) dxdt + \int_{t=0} u_0 \phi dx = 0 \\ \\ \iint\limits_{t>0} (v\phi_t + g(u,v)\phi_x) dxdt + \int_{t=0} v_0 \phi dx = 0 \end{cases} \tag{4}$$

hold for all test function $\phi \in C_0^1(R \times R^+)$.

To obtain a generalized solution of the Cauchy problem (1) and (3), a standard method is first to construct a sequence of approximated solutions of system (1) and then to consider the convergence of the sequence.

For instance, we add the viscosity terms to the right hand side of system (1) and study the following Cauchy problem for the parabolic system with the initial data (3):

$$u_t + f(u,v)_x = \epsilon u_{xx}, \quad v_t + g(u,v)_x = \epsilon v_{xx}. \tag{5}$$

For each fixed ϵ, suppose the solution (u^ϵ, v^ϵ) is uniformly bounded:

$$|u^\epsilon| \le M, \quad |v^\epsilon| \le M.$$

Then $f(u^\epsilon, v^\epsilon)$ and $g(u^\epsilon, v^\epsilon)$ are also uniformly bounded since f, g are continuous. Thus there exists a subsequence such that

$$(u^\epsilon, v^\epsilon, f(u^\epsilon, v^\epsilon), g(u^\epsilon, v^\epsilon)) \rightharpoonup (u, v, p, q)$$

weakly for some bounded functions $u(x,t)$, $v(x,t)$, $p(x,t)$ and $q(x,t)$.

Multiplying a test function ϕ to system (5), integrating on $R \times R^{+}$ and letting $\epsilon \to 0$, we have

$$\begin{cases} \iint\limits_{t>0} (u\phi_t + p(x,t)\phi_x)dxdt + \int_{t=0} u_0 \phi dx = 0 \\ \iint\limits_{t>0} (v\phi_t + q(x,t)\phi_x)dxdt + \int_{t=0} v_0 \phi dx = 0 \end{cases} \tag{6}$$

If one could prove that

$$p(x,t) = f(u(x,t), v(x,t)) \quad a.e.$$

and

$$q(x,t) = g(u(x,t), v(x,t)) \quad a.e.,$$

then clearly the pair of functions (u, v) is a generalized solution of the Cauchy problem (1) and (3).

How to prove the weak continuity of nonlinear functions $(f(u,v), g(u,v))$ with respect to the sequence (u^ϵ, v^ϵ) is the main content of the compensated compactness theory.

Why is this theory called Compensated Compactness? Roughly speaking, this term comes from the following fact:

If a sequence of functions satisfies

$$w^\varepsilon(x,t) \rightharpoonup w(x,t) \tag{7}$$

with either

$$\begin{cases} (w^\varepsilon)^2 + (w^\varepsilon)^3 \rightharpoonup w^2 + w^3 \text{ or} \\ (w^\varepsilon)^2 - (w^\varepsilon)^3 \rightharpoonup w^2 - w^3 \end{cases} \tag{8}$$

weakly as ε tends to zero, in general, $w^\varepsilon(x,t)$ is not compact. However, it is clear that any one weak compactness in

(8) can compensate for another to make the compactness of w^ε. In fact, if we add them together, we get

$$(w^\varepsilon)^2 \rightharpoonup w^2 \tag{9}$$

weakly as ε tends to zero, which combining with (7) implies the compactness of w^ε.

- The first application of the compensated compactness theory on the scalar equation

$$u_t + f(u)_x = 0$$

was obtained by L.Tartar in 1979.

A. The first application of the compensated compactness theory on system of two equations was obtained by R. DiPerna in 1983.

$\mathbf{A_1}$. System of One-dimensional Non-linear Elasticity

$$\begin{cases} u_t + f(v)_x = 0 \\ v_t + u_x = 0. \end{cases} \tag{10}$$

1. R. DiPerna: Arch. Rat. Mech. Anal., **82** (1983), 27-70. L^∞ solution.

(a) $f'(v) \geq c > 0$, (b) $v \cdot f''(v) > 0$, $\forall\, v \neq 0$;

2. P.-X. Lin, Trans. Am. Math. Soc., **329** (1992), 377-413 and

J. Shearer, Comm. Partial Diff. Eqs., **19** (1994), 1829-1877.

(b) $f'(v) \geq c > 0$, (b) $v \cdot f''(v) < 0$, $\forall\, v \neq 0$. L^p solution, $1 < p < \infty$.

OPEN PROBLEMS: $f'(v) \geq 0$.

$\mathbf{A_2}$. System of One-dimensional Isentropic Gas Dynamics in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = 0, \end{cases} \qquad (11)$$

with bounded measurable initial data

$$(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)), \quad (12)$$

where $\rho \geq 0$ is the density of gas, u the velocity, $P = P(\rho)$ the pressure satisfying $P'(\rho) \geq 0$. For the polytropic gas, P takes the special form $P(\rho) = c\rho^\gamma$, where $\gamma > 1$ and c is an arbitrary positive constant.

1. R. DiPerna: Commun. Math. Phys., **91** (1983), 1-30. $\gamma = 1 + \frac{2}{N}, N \geq 5$ odd;

2. Ding, Chen, Luo: Commun. Math. Phys., **121** (1989), 63-84. $\gamma \in (1, \frac{5}{3}]$;

3. Lions, Perthame and Tadmor: Commun. Math. Phys., **163** (1994), 415-431. $\gamma \geq 3$;

4. Lions, Perthame and Souganidis: Comm. Pure Appl. Math., **49** (1996), 599-638. $\gamma \in (\frac{5}{3}, 3)$;

$\mathbf{5}$. F.-M Huang and Z. Wang: SIAM J. Math. Anal., **34** (2003), 595-610. $\gamma = 1$;

$\mathbf{6}$. G.-Q. Chen and P. LeFloch: Arch. Rat. Mech. Anal., **166** (2003), 81-98.

Roughly speaking, $P(\rho)$ takes the form

$$P(\rho) = \rho^{\gamma}(1 + p(\rho)), \quad \gamma \in (1, 3),$$

with some restrictions on the function $p(\rho)$.

$\mathbf{7}$. Yunguang Lu: Differential Equations, **43** (2007), 130-138.

In this paper, we construct a sequence of regular hyperbolic systems

$$\begin{cases} \rho_t + (-2\delta u + \rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 - \delta u^2 + P_1(\rho, \delta))_x = 0, \end{cases} \tag{13}$$

to approximate the general system of isentropic gas dynamics (11), where

$$P_1(\rho, \delta) = \int_{2\delta}^{\rho} \frac{t - 2\delta}{t} P'(t) dt. \qquad (14)$$

System (13) is also nonstrictly hyperbolic since two eigenvalues are

$$\begin{cases} \lambda_1 = \frac{m}{\rho} - \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \\ \\ \lambda_2 = \frac{m}{\rho} + \frac{\rho - 2\delta}{\rho}\sqrt{P'(\rho)}, \end{cases} \qquad (15)$$

but both systems have the same entropy equation:

$$\eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2} \eta_{uu}.$$

First, for each fixed approximation parameter δ, we established the existence of entropy solutions for the Cauchy problem (13) with bounded initial date (12) in the optimal conditions on the pressure function $P(\rho)$:

$$P(\rho) \in C^2(0,\infty), \quad P'(\rho) > 0; \tag{16}$$

$$2P'(\rho) + \rho P''(\rho) > 0, \text{ for } \quad \rho > 0 \tag{17}$$

and

$$\int_c^\infty \frac{\sqrt{P'(\rho)}}{\rho} d\rho = \infty, \quad \int_0^c \frac{\sqrt{P'(\rho)}}{\rho} d\rho < \infty, \tag{18}$$

$\forall c > 0$.

Second, letting $\epsilon = o(\delta)$, we obtained a simple proof of the H^{-1}_{loc} compactness of weak entropy-entropy flux pairs of system (11) in the form $\eta(\rho,u) = \rho H(\rho,u)$

OPEN PROBLEMS: The limit as $\delta \to 0$?

B. The Existence of Generalized Solutions for Nonstrictly Hyperbolic System:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + (\frac{u^2}{2} + P(\rho))_x = 0, \end{cases} \tag{19}$$

where the function $P(\rho) = \frac{\gamma-1}{4}\rho^{\gamma-1}$ and $\gamma > 3$ is a constant.

System (19) was first derived by S. Earnshaw in 1858 for isentropic flow and has many other different physical backgrounds. By simple calculations, two eigenvalues of system (19) are

$$\lambda_1 = u - \frac{\gamma - 1}{2}\rho^{\frac{\gamma-1}{2}}, \quad \lambda_2 = u + \frac{\gamma - 1}{2}\rho^{\frac{\gamma-1}{2}} \tag{20}$$

with corresponding right eigenvectors

$$\begin{cases} r_1 = (1, -\frac{\gamma-1}{2}\rho^{\frac{\gamma-3}{2}})^T, \\ \\ r_2 = (1, \frac{\gamma-1}{2}\rho^{\frac{\gamma-3}{2}})^T; \end{cases} \tag{21}$$

the two corresponding Riemann invariants are

$$z = u - \rho^{\frac{\gamma-1}{2}}, \quad w = u + \rho^{\frac{\gamma-1}{2}}; \tag{22}$$

and

$$\begin{cases} \nabla\lambda_1 \cdot r_1 = -\frac{\gamma-1}{2}(\frac{\gamma+1}{2})\rho^{\frac{\gamma-3}{2}} \\ \nabla\lambda_2 \cdot r_2 = \frac{\gamma-1}{2}(\frac{\gamma+1}{2})\rho^{\frac{\gamma-3}{2}}. \end{cases} \tag{23}$$

Therefore, it follows from (20) that $\lambda_1 = \lambda_2$ at the line $\rho = 0$ at which the strict hyperbolicity fails to hold, and from (23) that both characteristic fields are linearly degenerate on $\rho = 0$ if $\gamma > 3$ and on $\rho = \infty$ if $1 < \gamma < 3$.

The study of the existence of global weak solutions for the Cauchy problem (19) was started by DiPerna for the case of $1 < \gamma < 3$ by using the Glimm's scheme method. Since system (19) has two different invariant regions, which induce two different L^∞ estimates as follows:

$$\rho \geq \min_{-\infty<x<\infty} \rho_0(x), \qquad \text{for } 1 < \gamma < 3 \tag{24}$$

and

$$0 \leq \rho \leq M, \quad |u| \leq M, \quad \text{for } \gamma > 3, \tag{25}$$

then the Glimm's scheme method works for the case of $1 < \gamma < 3$ since system (19) is strictly hyperbolic in the condition

$$\min_{-\infty < x < \infty} \rho_0(x) > 0. \tag{26}$$

However, for the case $\gamma > 3$, the strict hyperbolicity of system (19) fails since ρ could be zero at a finite time.

When we use the theory of compensated compactness to study system (19), the main difficulty is that system (19) has no a strictly convex entropy, which restrains the use of weak entropy-entropy flux pairs. To overcome this difficulty, in the paper

8. Y. Lu: Commun. Math. Phys., **150** (1992), 59-64,

we added a small perturbation δ to the nonlinear function $P(\rho)$:

$$P(\rho) = \int_0^\rho s^2(s+\delta)^{\gamma-3}ds$$

or

$$P(\rho) = \int_0^\rho s^2 e^s ds,$$

so that system (16) has a strictly convex entropy for any fixed $\delta > 0$ and hence, both strong and weak entropy-entropy flux pairs of the perturbation system of (16) satisfy the H^{-1} compactness condition. Therefore the existence of entropy solutions was obtained for this perturbation system.

9. Y. Lu, Proc. Roy. Soc. Edin. **124A** (1994), 341-352

10. Y. Lu and C. Klingenberg, Commun. Math. Phys., **187** (1997),327-340

$$P'(\rho) \geq \delta\rho^2.$$

In the paper

11. Y. Lu: Arch. Rat. Mech. Anal. **178**(2005), 287-299,

we studied the Cauchy problem for system (19) by using the kinetic formulation of systems of conservation laws developed by Lions, Perthame, Souganidis and Tadmor and obtained the main result as follows:

Theorem: The Cauchy problem (19) with bounded measurable initial data has a global bounded entropy solution.

One new idea is that we found a linear combination of strong and weak entropy satisfying the H^{-1} compactness condition.

One family of weak entropies of system (19) is given by

$$\eta_0(\rho,u)=\int_R g(\xi)G_0(\rho,\xi-u)d\xi, \quad (27)$$

two families of strong entropies of system (19) are given as follows

$$\eta_\pm(\rho,u)=\int_R g(\xi)G_\pm(\rho,\xi-u)d\xi, \quad (28)$$

where $g(\xi)$ is a smooth function with a compact support set in $(-\infty,\infty)$ and the fundamental solutions

$$\begin{cases} G_0(\rho,\xi-u)=[(w-\xi)(\xi-z)]_+^\lambda, \\ G_+(\rho,\xi-u)=(\xi-z)^\lambda(\xi-w)_+^\lambda, \\ G_-(\rho,\xi-u)=(w-\xi)^\lambda(z-\xi)_+^\lambda \end{cases} \quad (29)$$

and $\lambda=\frac{3-\gamma}{2(\gamma-1)}>-\frac{1}{2}$. Here we use the notation $x_+=max(0,x)$.

Lemma 1: For the viscosity solutions $(\rho^\epsilon,u^\epsilon)$ of system (19), if the entropy $\eta(\rho,u)$

of system (19) satisfy that

$$\eta_\rho(0,u)=0, \quad \frac{\partial^i \eta(\rho,u)}{\partial u^i} \quad i=0,1,2,3, \tag{30}$$

are bounded in $0 \le \rho \le M_1, |u| \le M_1$, then

$$\eta(\rho^\epsilon(x,t),u^\epsilon(x,t))_t + q(\rho^\epsilon(x,t),u^\epsilon(x,t))_x \tag{31}$$

is compact in $H^{-1}_{loc}(R\times R^+)$ as ϵ and tends to zero, where q is the entropy flux of system (19) associated with η.

Lemma 2: For the viscosity solutions $(\rho^\epsilon, u^\epsilon)$ of system (19),

$$\eta_j(\rho^\epsilon(x,t),u^\epsilon(x,t))_t + q_j(\rho^\epsilon(x,t),u^\epsilon(x,t))_x \tag{32}$$

are compact in $H^{-1}_{loc}(R\times R^+)$ as ϵ tends to zero, where $j=1,2,3$ and

$$C=\frac{(\gamma-3)\int_0^\infty (s+2)^{\lambda-1}s^\lambda ds}{2\int_{-1}^1(1-s^2)^\lambda ds}>0, \tag{33}$$

$$\eta_1 = \eta_+ + C\eta_0, \quad \eta_2 = \eta_- + C\eta_0, \quad (34)$$

$$\eta_3 = \eta_+ - \eta_-, \quad (35)$$

$\eta_\pm, \eta_0$ being given by (27), (28) and q_j are corresponding entropy fluxes of η_j.

We only give the proof of Lemma 2 for (η_1, q_1). A similar treatment gives the proof for $(\eta_j, q_j), j = 2, 3$.

Let $\tau = \xi - w$. Then

$$\begin{aligned} \eta_+(\rho, u) &= \int_w^\infty g(\xi)(\xi - z)^\lambda (\xi - w)^\lambda d\xi \\ &= \int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau \end{aligned} \quad (36)$$

and hence

$$\begin{aligned} &\eta_+(\rho, u)_\rho \\ &= (\int_0^\infty g'(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau)\theta\rho^{\theta-1} \\ &+(\int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^{\lambda-1}\tau^\lambda d\tau) 2\lambda\theta\rho^{\theta-1}. \end{aligned} \quad (37)$$

Since $-1 < 2\lambda < 0$, the first term on the right-hand side tends to zero as ρ tends zero. Let $\tau = \rho^\theta s$ in the second part of the right hand-side of (37). Then

$$\eta_+(\rho, u)_\rho$$

$$= \int_0^\infty g'(\tau + w)(\tau + 2\rho^\theta)^\lambda \tau^\lambda d\tau \theta \rho^{\theta-1}$$

$$+ \int_0^\infty g(\rho^\theta s + w)(s + 2)^{\lambda-1} s^\lambda ds 2\lambda\theta, \tag{38}$$

since $(\rho^\theta)^{2\lambda+1}\rho^{-1} = 1$. Thus

$$\eta_+(\rho, u)_\rho|_{\rho=0} = 2\lambda\theta g(u) \int_0^\infty (s+2)^{\lambda-1} s^\lambda ds. \tag{39}$$

Similarly

$$\eta_0(\rho, u) = \rho \int_{-1}^1 g(u + \rho^\theta s)(1 - s^2)^\lambda ds \tag{40}$$

and hence

$$\eta_0(\rho,u)_\rho$$

$$= \int_{-1}^{1} g(u+\rho^\theta s)(1-s^2)^\lambda ds \qquad (41)$$

$$+\theta\rho^\theta \int_{-1}^{1} g'(u+\rho^\theta s)(1-s^2)^\lambda ds$$

or

$$\eta_0(\rho,u)_\rho|_{\rho=0} = g(u)\int_{-1}^{1}(1-s^2)^\lambda ds. \quad (42)$$

Combining (39) and (42), we have

$\eta_1(\rho,u)_\rho|_{\rho=0} = 0$. It is easy to see that η_1 is smooth on the variable u, hence the proof of Lemma 2 is ended by using Lemma 1.